THE KERNEL OF THE COMPOSITION OF CHARACTERISTIC FUNCTION GAMES[1]

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ABSTRACT

The structure of the kernel of a composition of two games is investigated; a comparison with the results for Von-Neumann and Morgenstern solutions is included.

1. **Introduction.** In this paper we investigate the kernel of the composition, in the sense of von Neumann and Morgenstern, of two games. Sections 2, 3, and 4 contain the necessary definitions, preparatory lemmas and an example which shows that the phenomenon of transfer occurs in the kernel. In Section 5 we give a complete description of the kernel of a composition and obtain bounds on the transfer. The case of composition of two ordinary three-person games is studied in full detail in Section 6. In Section 7 we apply the results of Section 5 to constant-sum games; it turns out that the results obtained are similar to the results of von Neumann and Morgenstern for solutions.

2. **Definitions.** Let $N = \{1, 2, .., n\}$ be a set with n members. A *characteristic function* is a non-negative real function v defined on the subsets of N which satisfies

$$(2.1) \qquad v(\phi) = 0, \text{ and } v(\{i\}) = 0, \text{ for } i = 1, \cdots, n.$$

The pair $(N; v)$ is an *n-person game*. The members of N are called *players*. Subsets of N are called *coalitions*. The game $(N; v)$ is an *ordinary game* if

$$(2.2) \qquad v(S) \leqq v(N), \quad \text{for all} \quad S \subset N.$$

Let $G = (N; v)$ be an n-person game. An *individually rational payoff vector* (i.r.p.v.) is an n-tuple $x = (x_1, \cdots, x_n)$ of real numbers which satisfies:

Received June 21, 1965

(1) The research described in this paper was partially supported by the U.S. Office of Naval Research under Contract Number N62558-4355.

(2.3) $$x_i \geqq 0, \quad i = 1, \cdots, n \quad \text{(individual rationality)},$$

and

(2.4) $$\sum_{i=1}^{n} x_i = v(N).$$

The set of all the i.r.p.v's is denoted by $A(G)$.

Let i and j be two different players. We denote by $\mathscr{T}_{ij}$ the set of all the coalitions which contain player i but do not contain player j; i.e.,

(2.5) $$\mathscr{T}_{ij} = \{S\colon\ S \subset N, \quad i \in S \quad \text{and} \quad j \notin S\}.$$

Let x be an i.r.p.v. and let S be an arbitrary coalition. The *excess* of S with respect to x is(2);

(2.6) $$e(S, x) = v(S) - \sum_{i \in S} x_i$$

The *maximum surplus* of i over j with respect to x is

(2.7) $$s_{ij}(x) = \max\{e(R, x) : R \in \mathscr{T}_{ij}\}$$

i is said to *outweigh* j with respect to x if

(2.8) $$s_{ij}(x) > s_{ji}(x) \quad \text{and} \quad x_j > 0.$$

x is *balanced* if there exists no pair of players h and k such that h outweighs k. The *kernel*(3),(4) of G, $\mathscr{K}(G)$, is the set of all balanced i.r.p.v's.

Let c be a real number which satisfies the inequality $c \leqq v(N)$. We define a new game $G(c) = (N; v_c)$, with the same set of players N as in G, whose characteristic function v_c is defined by

(2.9) $$v_c(S) = \begin{cases} v(S), & S \neq N \\ v(N) - c, & S = N. \end{cases}$$

The *core*(5) of the game G is the set of all i.r.p.v.'s x which satisfy

(2.10) $$e(S, x) \leqq 0, \quad \text{for all} \quad S \subset N.$$

The *strict core* is the set of all i.r.p.v.'s which satisfy

(2.11) $$e(S, x) < 0, \text{ for } S \subset N,\ S \neq \varnothing \text{ and } S \neq N.$$

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(2) $e(\phi, x)$ is taken as equal to zero.

(3) The reader is referred to [2] for a comprehensive introduction to the kernel theory, and to [4] for recent developments.

(4) For the coalition structure $\{N\}$ (see [2]),

(5) See [7, page 1] for the origin of this concept.

3. **The kernel of the composition of two games.** Let $G_1 = (N_1; v_1)$ and $G_2 = (N_2; v_2)$ be two games with *disjoint* sets of players, (i.e., $N_1 \cap N_2 = \emptyset$). The composition(6) of G_1 and G_2 is the game $G = (N; v)$ which satisfies

$$N = N_1 \cup N_2, \tag{3.1}$$

and

$$v(S) = v_1(S \cap N_1) + v_2(S \cap N_2), \text{ for all } S \subset N. \tag{3.2}$$

EXAMPLE 3.1. Let G_1 and G_2 be two three-person constant-sum games in (0, 1) normalization, with disjoint sets of players. In G, the composition of G_1 and G_2, each pair of players of G_1, or of G_2, are symmetric. Hence, by [4] Theorem 9.3, the i.r.p.v's of the kernel of G are of the form $(x, x, x, 2/3 - x, 2/3 - x, 2/3 - x)$, where x is limited by the requirement $\max(1-2x, 2/3) = \max(2x - 1/3, 2/3)$, i.e., $1/6 \leqq x \leqq 1/2$.

We see that $\mathscr{K}(G)$ contains i.r.p.v's outside the cartesian product $\mathscr{K}(G_1) \times \mathscr{K}(G_2) = (1/3, 1/3, 1/3, 1/3, 1/3, 1/3)$. Thus, as in the set of von Neumann and Morgenstern solutions of a composition of games, the phenomenon of transfer occurs; it will be the subject of our following investigations.

4. **Auxiliary lemmas.** We shall prove in this section two simple results on balanced i.r.p.v's, which will be needed in the following section.

Let $G = (N; v)$ be an n-person game and $c \leqq v(N)$. For $x \in A(G(c))$ (see (2.9)) the following functions are defined

$$f(x) = \max\{e(S, x) : S \subset N,\ S \neq \emptyset \text{ and } S \neq N\}, \tag{4.1}$$

$$g_i(x) = \max\{\max\{e(S, x) : i \in S,\ S \neq N\}, c\} \text{ and} \tag{4.2}$$

$$h_i(x) = \max\{e(S, x) : i \notin S\}, \tag{4.3}$$

where $i = 1, \cdots, n$. Observe that $h_i(x)$ is non-negative since $e(\emptyset, x) = 0$.

The first lemma is concerned with unmodified ordinary games.

LEMMA 4.1. *If G is an ordinary game (see (2.2)) then $h_i(x) \geqq g_i(x)$ for all $x \in \mathscr{K}(G)$, for all $i \in N$.*

Proof. Assume, per absurdum, that there exist $x \in \mathscr{K}(G)$ and $i \in N$ such that $g_i(x) > h_i(x)$. It follows that there exists a coalition S which satisfies $i \in S$ and $g_i(x) = e(S, x) > h_i(x)$. Let $j \in N - S$. $s_{ij}(x) \geqq e(S, x) > h_i(x) \geqq s_{ji}(x)$. Since x is balanced, $x_j = 0$. Hence

$$e(S, x) = v(S) - \sum_{k \in S} x_k = v(S) - \sum_{k \in N} x_k \leqq v(N) - \sum_{k \in N} x_k = 0.$$

Thus $g_i(x) \leqq 0$. But we have assumed that $g_i(x) > h_i(x) \geqq 0$. Hence, our assumption cannot hold, and the proof is complete

(6) See [5], Chapter IX, for the origin of this definition.

The second lemma applies to modified games.

LEMMA 4.2. *Let* $x \in \mathscr{K}(G(c))$ *and* $i \in N$. *If* $f(x) \geqq 0$ *and* $x_i > 0$ *then* $g_i(x) \geqq h_i(x)$.

Proof. $f(x) \leqq \max(g_i(x), h_i(x))$. Assume $h_i(x) > g_i(x)$; then $f(x) \leqq h_i(x)$. Our assumption $f(x) \geqq 0$ implies that $f(x) \geqq h_i(x)$. Thus $f(x) = h_i(x)$. It follows that $h_i(x) = e(S, x)$ for a coalition S which satisfies $S \neq \varnothing$ and $i \notin S$. Let $j \in S \cdot s_{ji}(x) \geqq e(S, x) > g_i(x) \geqq s_{ij}(x)$. Thus j outweighs i, which is impossible. Hence our assumption $h_i(x) > g_i(x)$ cannot hold.

5. **The Structure of the kernel of a composition of two games.** Let $G_1 = (N_1; v_1)$ and $G_2 = (N_2; v_2)$ be two *ordinary* games with disjoint sets of players, and let $G = (N; v)$ be the composition of G_1 and G_2. An i.r.p.v. $z \in A(G)$ is a real function defined on N which satisfies (2.3) and (2.4). The restrictions of z to N_1 and to N_2 will be denoted by x and y respectively. The amount of transfer exhibited by z is given by

$$t = t(z) = v_1(N_1) - \sum_{i \in N_1} x_i \tag{5.1}$$

The games $G_1(t)$ and $G_2(-t)$ (see (2.9)) are related in a natural way to z. Clearly $x \in A(G_1(t))$ and $y \in A(G_2(-t))$. In denoting function of x and y we adopt the following convention: functions depending only on $x[y]$ will bear the superscript "1" ["2"]. Thus, e.g., if $i, j \in N_1$ then $s^1_{ij}(x)$ is the maximum surplus of i over j with trespect to x (see (2.7)) in the game $G_1(t)$.

LEMMA 5.1. *Let* $z \in A(G)$ *and* $i, j \in N_1[i, j \in N_2]$. *i outweighs j with respect to z (see* (2.8)) *in G, if and only if i outweighs j with respect to* $x[y]$ *in* $G_1(t)[G_2(-t)]$, *where t is defined by* (5.1).

Proof. Clearly it is sufficient to consider only the case $i, j \in N_1$.

$$\begin{aligned} s_{ij}(z) &= \max\{e(S, z) : S \in \mathscr{T}_{ij}\} = \max\{e(S, z) : S \cap N_1 \in \mathscr{T}_{ij}\} \\ &= \max\{e(S, z) + e(R, z) : S \subset N_1, S \in \mathscr{T}_{ij} \text{ and } R \subset N_2\} \\ &= \max\{e^1(S, x) : S \subset N_1, S \in \mathscr{T}_{ij}\} + \max\{e(R, z) : R \subset N_2\} \\ &= s^1_{ij}(x) + \max\{e(R, z) : R \subset N_2\} \end{aligned}$$

Similarly $s_{ji}(z) = s^1_{ji}(x) + \max\{e(R, z) : R \subset N_2\}$. Hence $s^1_{ij}(x) > s^1_{ji}(x)$ if and only if $s_{ij}(z) > s_{ji}(z)$. Since $x_j = z_j$, the proof follows.

COROLLARY 5.2. *If* $z \in \mathscr{K}(G)$ then $x \in \mathscr{K}(G_1(t))$ *and* $y \in \mathscr{K}(G_2(-t))$.

LEMMA 5.3. *If* $z \in A(G)$, $i \in N_1$ *and* $j \in N_2$ *then*

$$s_{ij}(z) = g^1_i(x) + h^2_j(y) \qquad (s_{ji}(z) = g^2_j(y) + h^1_i(x)).$$

Proof. Again only the first half of the lemma will be proved.

$s_{ij}(z) = \max\{e(S,z) : S \in \mathcal{T}_{ij}\} = \max\{e(S,z) + e(R,z) : i \in S, S \subset N_1,$ and $R \subset N_2, j \notin R\} = \max\{e(S,z) : i \in S, S \subset N_1\} + \max\{e(R,z) : j \notin R, R \subset N_2\}$ $= g_i^1(x) + h_j^2(y)$, (see (4.2) and (4.3)).

We remark that $g_i^1(x)\,[h_j^2(y)]$ is computed here with respect to $G_1(t)\,[G_2(-t)]$.

THEOREM 5.4. $\mathcal{K}(G) \cap [A(G_1) \times A(G_2)] = \mathcal{K}(G_1) \times \mathcal{K}(G_2)$.

Proof. Corollary 5.2 implies that $\mathcal{K}(G_1) \times \mathcal{K}(G_2) \supset \mathcal{K}(G) \cap [A(G_1) \times A(G_2)]$. Let now $x \in \mathcal{K}(G_1)$, $y \in \mathcal{K}(G_2)$ and $z = (x, y)$. *If* i and j both belong to N_1, or both belong to N_2, then, by lemma 5.1 j does not outweigh i with respect to the game G. So let $i \in N_1$ with $x_i > 0$ and $j \in N_2$. By Lemma 5.3 $s_{ij}(z) = g_i^1(x) + h_j^2(y)$ and $s_{ji}(z) = g_j^2(y) + h_i^1(x)$.We shall show that $g_i^1(x) \geqq h_i^1(x)$. To prove this we observe that if $f^1(x) < 0$, (see (4.1)) then $g_i^1(x) = h_i^1(x) = 0$, and if $f^1(x) \geqq 0$, then by Lemma 4.2, $g_i^1(x) \geqq h_i^1(x)$. By Lemma 4.1 $h_j^2(y) \geqq g_j^2(y)$. Thus $s_{ij}(z) \geqq s_{ji}(z)$ and j does not outweigh i. Clearly we can interchange N_1 and N_2 in the above argument. Thus the proof is complete.

The following example shows that our assumption that both G_1 and G_2 are ordinary games is essential for the validity of Theorem 5.4. But since constant-sum(7), superadditive(7), or even monotonic(7) games are ordinary games, (2.2) is not too restricting assumption.

EXAMPLE 5.5. Let us consider the following two games: $(\{1,2\}; u_1)$, where $u_1(\{1,2\}) = 1$, and $(\{a,b,c\}; u_2)$, where $u_2(\{a,b\}) = 1 + \varepsilon$, $0 < \varepsilon < 1$, $u_2(\{a,c\}) = 0$, $u_2(\{b,c\}) = 0$, and $u_2(\{a,b,c\}) = 1$. The second is not an ordinary game. The kernel of the first game is $(\frac{1}{2}, \frac{1}{2})$, and the kernel of the second is $(\frac{1}{2}, \frac{1}{2}, 0)$. The kernel of the composition of these two games is

$$\left(\frac{1}{2} - \frac{\varepsilon}{4}, \frac{1}{2} - \frac{\varepsilon}{4}, \frac{1}{2} + \frac{\varepsilon}{4}, \frac{1}{2} + \frac{\varepsilon}{4}, 0\right).$$

We proceed now to determine the structure of $\mathcal{K}(G)$.

LEMMA 5.6. *Let* $z \in \mathcal{K}(G)$. *If* $t < 0$ *then* $f^1(x) \geqq 0$.

Proof. Suppose, per absurdum, that $f^1(x) < 0$. Choose $i \in N_1$ who satisfies $x_i > 0$. This is possible since $t < 0$. Our assumptions imply that $g_i^1(x) < 0$.

(7) A game $(M; u)$ is constant-sum if $u(S) + u(M - S) = u(M)$, for all $S \subset M$. It is superadditive if $u(S) + u(T) \leqq u(S \cup T)$, for all pairs of disjoint coalitions S and T. $(M; u)$ is monotonic if $u(S) \leqq u(T)$ whenever $S \subset T$.

Let $S \subset N_2$ be a coalition with the greatest excess, i.e., $e(S,z) \geqq e(R,z)$ for all $R \subset N_2$. $t < 0$ implies that $S \neq \emptyset$. Let $j \in S$.

$$s_{ji}(z) = g_j^2(y) + h_i^1(x) \geqq g_j^2(y) \geqq h_j^2(y) > h_j^2(y) + g_i^1(x) = s_{ij}(z).$$

Thus j outweighs i, contradicting our assumption $z \in \mathscr{K}(G)$. Hence $f^1(x) \geqq 0$ must hold.

COROLLARY 5.7. *If $z \in \mathscr{K}(G)$ and $t > 0$ then $f^2(y) \geqq 0$.*

LEMMA 5.8. *If $z \in \mathscr{K}(G)$ and $t > 0$ then $h_i^1(x) \geqq g_i^1(x)$ for all $i \in N_1$.*

Proof. Assume that there exists an i such that $g_i^1(x) > h_i^1(x)$. Let $j \in N_2$ be a player who gets a positive payment in z, i.e., $z_j > 0$. Since $t > 0$ such a player exists. Since $-t < 0$, $G_2(-t)$ is an ordinary game. By Corollary 5.2 $y \in \mathscr{K}(G_2(-t))$. Hence Lemma 4.1 implies that $h_j^2(y) \geqq g_j^2(y)$. Combining the above inequalities we have

$$s_{ij}(z) = g_i^1(x) + h_j^2(y) > h_i^1(x) + g_j^2(y) = s_{ji}(z)$$

i.e., i outweighs j with respect to z, contradicting our assumption that $z \in \mathscr{K}(G)$. Hence $h_i^1(x) \geqq g_i^1(x)$ must hold for all $i \in N_1$.

COROLLARY 5.9. *If $z \in \mathscr{K}(G)$ and $t < 0$ then $h_i^2(y) \geqq g_i^2(y)$ for all $i \in N_2$.*

COROLLARY 5.10. *If $z \in \mathscr{K}(G)$ and $t > 0[t < 0]$ then $f^1(x) \geqq t[f^2(y) \geqq -t]$.*

Proof. Suppose $f^1(x) < t$. If $i \in N_1$ then $g_i^1(x) = t$. Hence

$$h_i^1(x) \leqq \max(0, f^1(x)) < g_i^1(x),$$

which contradicts Lemma 5.8. Thus $f^1(x) \geqq t$ must hold.

Although we do not use this result, it is included since it provides an interesting bound on the transfer.

LEMMA 5.11. *Let $v_1(N_1) \geqq t > 0$, $x \in \mathscr{K}(G_1(t))$, $y \in \mathscr{K}(G_2(-t))$ and $z = (x,y)$. If $h_i^1(x) \geqq g_i^1(x)$ for all $i \in N_1$ and $f^2(y) \geqq 0$, then $z \in \mathscr{K}(G)$.*

Proof. If i and j both belong to N_1, or both belong to N_2, then by Lemma 5.1 j does not outweigh i. We have to distinguish two additional possibilities:

(a) $i \in N_1$ and $j \in N_2$.

If $x_i = 0$ then clearly j does not outweigh i. If $x_i > 0$ we claim that

$$g_i^1(x) \geqq h_i^1(x).$$

To see this observe that $h_i^1(x) \geqq g_i^1(x) \geqq t > 0$; hence $f^1(x) \geqq t > 0$. Thus, by Lemma 4.2, $g_i^1(x) \geqq h_i^1(x)$. In addition we have, since $G_2(-t)$ is an ordinary game, $h_j^2(y) \geqq g_j^2(y)$ (*see* Lemma 4.1). Combining the above inequalities

$$s_{ij}(z) = g_i^1(x) + h_j^2(y) \geqq h_i^1(x) + g_j^2(y) = s_{ji}(z)$$

and thus j cannot outweigh i.

(b) $i \in N_2$ and $j \in N_1$.

If $y_i > 0$ then, since $f^2(y) \geqq 0$, $g_i^2(y) \geqq h_i^2(y)$, (see Lemma 4.2). Adding our assumption that $h_j^1(x) \geqq g_j^1(x)$

$$s_{ij}(z) = g_i^2(y) + h_j^1(x) \geqq h_i^2(y) + g_j^1(x) = s_{ji}(z).$$

Hence j cannot outweigh i.

COROLLARY 5.12. *Let* $0 > t \geqq -v_2(N_2)$, $x \in \mathscr{K}(G_1(t))$, $y \in \mathscr{K}(G_2(-t))$ *and* $z = (x, y)$. *If* $h_i^2(y) \geqq g_i^2(y)$ *for all* $i \in N_2$ *and* $f^1(x) \geqq 0$, *then* $z \in \mathscr{K}(G)$.

In order to be able to formulate and interpret our results in a satisfactory way we need the following:

DEFINITION 5.13. Let $H = (M; u)$ be an ordinary game. For $c \leqq u(M)$ we define

$$\mathscr{K}^*(H(c)) = \begin{cases} \{x : x \in \mathscr{K}(H(c)) \text{ and } f(x) \geqq 0\}, & c < 0 \\ \mathscr{K}(H), & c = 0 \\ \{x : x \in \mathscr{K}(H(c)) \text{ and } h_i(x) \geqq g_i(x) \text{ for all } i \in M\}, & u(M) \geqq c > 0 \end{cases}$$

For $c < 0$ $\mathscr{K}^*(H(c))$ consists of the part of $\mathscr{K}(H(c))$ which is *outside* the strict core of $H(c)$ (see (2.11)). When $c = 0$ $\mathscr{K}^*(H(c))$ coincides with $\mathscr{K}(H)$. When $u(M) \geqq c > 0$ the interpretation is less intuitive. It seems that in this case $\mathscr{K}^*(H(c))$ contains those i.r.p.v's of $\mathscr{K}(H(c))$ which preserve certain properties which are characteristic of i.r.p.v's of kernels of ordinary games (see Lemma 4.1). We emphasize that when $u(M) \geqq c > 0$, $H(c)$ is *not* and ordinary game, and $\mathscr{K}^*(H(c))$ may be strictly contained in $\mathscr{K}(H(c))$, or even be empty.

We now denote

(5.2) $\quad D(H) = \{c : \mathscr{K}^*(H(c)) \neq \emptyset\}$,

(5.3) $\quad \bar{D}(H) = \{c : 0 \leqq c \leqq u(M) \wedge (\exists x)(x \in \mathscr{K}(H(c)) \wedge h_i(x) \geqq g_i(x), i \in M)\}$

(5.4) $\quad \underline{D}(H) = \{c : c \leqq 0 \wedge (\exists x)(x \in \mathscr{K}(H(c)) \wedge f(x) \geqq 0)\}$.

By Lemma 4.1 $0 \in \bar{D}(H)$; hence $D(H) = \bar{D}(H) \cup \underline{D}(H)$.

LEMMA 5.14. *$\bar{D}(H)$ and $\underline{D}(H)$ are compact.*

Proof. $\bar{D}(H)$ is clearly bounded. To see that $\underline{D}(H)$ is bounded observe that if $c < -(|M|^2 - 1)u(M)$ then $\mathscr{K}(H(c))$ is contained in the strict core of [8] $H(c)$ (here $|M|$ denotes the number of players in M). Since $\mathscr{K}(H(c))$ is uniformly

[8] Let $x \in \mathscr{K}(H(c))$ and let i be a player who gets a maximum payoff in x. $x_i > |M| u(M)$. Let $j \neq i$. If $S \in \mathscr{T}_{ij}$ then $e(S, x) < -(|M| - 1)u(M)$. Since $s_{ij}(x) \geqq s_{ji}(x)$, we must have $x_j > u(M)$. Thus it is clear that x must be in the strict core of $H(c)$.

bounded when c is restricted to a bounded set, and is an upper semicontinuous function(9) of c, both $\bar{D}(H)$ and $\underline{D}(H)$ are closed.

Lemma 5.14 renders the following definitions natural

$$\bar{c}(H) = \max\{c : c \in \bar{D}(H)\} \tag{5.5}$$

$$\underline{c}(H) = \min\{c : c \in \underline{D}(H)\} \tag{5.6}$$

LEMMA 5.15. *$\bar{D}(H)$ $[\underline{D}(H)]$, and hence $\bar{c}(H)[\underline{c}(H)]$, can be effectively determined.*

Proof. To determine $\bar{D}(H)$ one has to eliminate x out of the formula

$$(\exists x)(x \in \mathscr{K}(H(c)) \wedge h_i(x) \geqq g_i(x),\ i \in M) \tag{5.7}$$

The nature of the inequalities involved enables us to carry out such an elimination (See [3] for a detailed account of such a procedure). Thus (5.7) is equivalent to a set of linear inequalities (with integral coefficients) in c and the $u(S)$, $S \subset M$. Adding the inequality $0 \leqq c \leqq u(M)$, we can represent $\bar{D}(H)$ as a finite union of closed intervals(10) with end points which are linear functions (with rational coefficients) *of the values* $u(S)$, $S \subset M$.

LEMMA 5.16. *If H is a two person game then $D(H) = \{0\}$.*

Proof. $\bar{D}(H) = \{0\}$ and $\underline{D}(H) \subset \{0\}$ (see (5.3), (5.4)).

We now return to the description of the kernel of G, the composition of the ordinary games G_1 and G_2.

THEOREM 5.17. *Let G be a composition of ordinary games G_1 and G_2; then the kernel of G is given by* (11)

$$\mathscr{K}(G) = \cup\{\mathscr{K}^*(G_1(t)) \times \mathscr{K}^*(G_2(-t)) : t \in D(G_1) \cap -D(G_2)\} \tag{5.8}$$

(see Definition 5.13 and (5.2)).

Proof. Corollary 5.2., Theorem 5.4., Lemma 5.6, Corollary 5.7, Lemma 5.8, Corollary 5.9, Lemma 5.11 and Corollary 5.12.

COROLLARY 5.18. *The maximum amount, according to $\mathscr{K}(G)$, which is given by N_1 to N_2, is*

$$\bar{t} = \min(\bar{c}(G_1), -\underline{c}(G_2)) \text{ (see (5.5), (5.6))}, \tag{5.9}$$

and the maximum amount which is given by N_2 to N_1, is

$$\underline{t} = \min(\bar{c}(G_2), -\underline{c}(G_1)). \tag{5.10}$$

(9) See [1] Theorem 3.1.

(10) We conjecture that $\bar{D}(H)$ $[\underline{D}(H)]$, and hence $D(H)$, is an interval.

(11) $-D(G_2) = \{c : -c \in D(G_2)\}$. See also the footnote above, from which it would follow that the range of t is an interval.

COROLLARY 5.19. *If G_1 (or G_2) is a two person game then*

$$\mathscr{K}(G) = \mathscr{K}(G_1) \times \mathscr{K}(G_2) \tag{5.11}$$

Proof. Theorem 5.17 and Lemma 5.16.

We remark that the results for non-ordinary games may be drastically different, and many degeneracies may occur.

6. **Composition of three-person games,** The first non-trivial case of a kernel of a composition of games is the case of two three-person games (see Example 3.1 and Corollary 5.19), which will be studied in this section.

Let $G = (N; v)$ be an ordinary 3-person game. To compute $\underline{c}(G)$ we recall that the kernel of a three-person game(12) consists of a unique point ([2], Section 4) which is contained in the strict core, if the strict core is not empty ([4, Theorem 5.4]). The condition that the strict core of the game $G(c)$, $c \leqq 0$, is empty, is

$$v(N) - c \leqq \frac{v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\})}{2} \tag{6.1}$$

Hence, if $\underline{D}(G)$ is not empty, i.e., (6.1) is satisfied for a non-positive value of c, then

$$\underline{c}(G) = v(N) - \frac{v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})}{2} \tag{6.2}$$

To compute $\bar{c}(G)$ observe that if $c > 0$, $c \in \bar{D}(G)$, and $x \in \mathscr{K}(G(c))$, then $g_i(x) = \max(e(\{i,j\},x),\ e(\{i,k\},x), c) > 0$, where $N = \{i,j,k\}$; hence $h_i(x) = e(\{j,k\},x)$. Since $h(x) \geqq g_i(x)$, $i \in N$, we have

$$e(\{i,j\},x) = e(\{i,k\},x) = e(\{j,k\},x) \geqq c \tag{6.3}$$

A straightforward computation shows that there exists an i.r.p.v. $x \in A(G(c))$ that satisfies (6.3) if and only if

$$c \leqq v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) - 2v(N). \tag{6.4}$$

Hence

$$\bar{c}(G) = \max(0, v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) - 2v(N)) \tag{6.5}$$

Recalling that $\underline{c}(G) < 0$ is a necessary and sufficient condition for G to have an empty core, we have

THEOREM 6.1. *Let G be an ordinary three-person game. If the core of G is empty then $D(G)$ consists of the interval*

$$\left[v(N) - \frac{v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})}{2},\ v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) - 2v(N)\right];$$

if the core of G is not empty then $D(G) = \{0\}$.

(12) Not necessarily ordinary.

Corollary 6.2. *Let G_1 be an ordinary three-person game with a non-empty core. If G_2 is an ordinary game, then the kernel of the composition of G_1 and G_2 is equal to $\mathscr{K}(G_1) \times \mathscr{K}(G_2)$.*

Proof. Theorems 5.17 and 6.1.

We remark that Corollary 6.2 cannot be generalized to games with an arbitrary number of players. The counter examples that we have at present require long computations, and therefore will not be given here

7. **Comparison with the results for solution theory.** In this section we shall compare the results we have for the kernel of a composition with the results for the von Neumann Morgenstern solutions ([5], Chapter IX). In order to carry out our program we need the following.

Lemma 7.1. *Let $H = (M;u)$ be a superadditive*(13) *constant-sum game, and let $0 \leqq c \leqq u(M)$. If $x \in \mathscr{K}(H(c))$ then $h_i(x) \geqq g_i(x)$ for all $i \in M$.*

Proof. *Suppose there exists an $i \in M$ such that $g_i(x) > h_i(x)$.* Since $u(M - \{j\}) = u(M)$ for all $j \in M$, there is a coalition $S \neq M$ such that $g_i(x) = e(S,x)$ (see (4.2)). Furthermore, $f(x) = g_i(x)$ (see also (4.1), (4.3)). Let $j \in M - S$. $s_{ij}(x) \geqq e(S,x) > h_i(x) \geqq s_{ji}(x)$. Hence $x_j = 0$. Thus

$$g_i(x) = e(S,x) = u(S) - \sum_{k \in S} x_k = u(S) - \sum_{k \in M} x_k \leqq u(M) - (u(M) - c) = c\,.$$

Let now $h \in S$ *with*(14) $x_h > 0$ and $R = M - \{h\}$.

$$e(R,x) = u(R) - \sum_{k \in R} x_k = u(M) - \sum_{k \in R} x_k > c\,.$$

Since $f(x) \geqq e(R,x)$, we have a contradiction and the proof is complete.

Corollary 7.2. *Let $H = (M;u)$ be a superadditive constant-sum game. For $0 \leqq c \leqq u(M) \mathscr{K}(H(c)) = \mathscr{K}^*(H(c))$* (see Definition 5.13); *furthermore , $\bar{D}(H)$ is equal to the interval* $[0, u(M)]$ (see (5.3)).

We now recall a definition of von Neumann and Morgenstern ([5], p. 369). Let $H = (M;u)$ be a superadditive constant-sum game. An i.r.p.v. in $A(H(c))$ is (*fully*) *detached* if it belongs to the (strict) core of $H(c)$. We remark that (fully) detached i.r.p.v.'s exist only if $c \leqq 0$ $[c < 0]$.

Let now $G_1 = (N_1;v_1)$ and $G_2 = (N_2;v_2)$ be two superadditive constant-sum games with disjoint sets of players, and let G be the composition of G_1 and G_2. Combining Theorem 5.17 and Corollary 7.2 we get (15)

(13) This assumption is not used in the proof.
(14) If no such player exists then $c = u(M)$ and the proof is immediate.
(15) See Section 5 for the notation.

COROLLARY 7.3. *An i.r.p.v. $z \in \mathscr{K}(G)$ if and only if the following three conditions are statisfied*:

(7.1) $x \in \mathscr{K}(G_1(t))$ *and* $y \in \mathscr{K}(G_2(-t))$

(7.2) *x and y are not fully detached*

(7.3) $-v_2(N_2) \leqq t \leqq v_1(N_1)$

Corollary 7.3 is to be compared with the results of von Neumann and Morgenstern for solutions ([5], p. 401). In our opinion the similarity deserves notice, since it is seen that the i.r.p.v's of the kernel and the von Neumann Morgenstern solutions obey the same composition rules. The comparison seems to us legitimate since both the balanced payoffs and the solutions specify the payments to the players(16).

We remark that the situation for non constant-sum games may be different. To take an extreme case let us consider the following example.

EXAMPLE 7.4. Let $G_1 = (\{1,2,3\};v_1)$ be a constant-sum three-person game in (0,1) normalization and let $G_2 = (\{a,b,c\};v_2)$ be given by $v_2(\{a,b\}) = v_2(\{a,c\}) = v_2(\{a,b,c\}) = 1$, and $v_2(\{b,c\}) = 0$. If G is the composition of G_1 and G_2 then by Corollary 6.2, $\mathscr{K}(G) = \mathscr{K}(G_1) \times \mathscr{K}(G_2)$, while the solutions of G are obtained by forming all possible cartesian products of solutions of $G_1(t)$ and $G_2(-t)$ for $-\frac{1}{2} \leqq t \leqq 0$.

A discussion of the solutions of a composition of two general sum games(17) can be found in [6]. The main result is:

THEOREM 7.5. *Let $G = (N;v)$ be a composition of two (general sum) games $G_1 = (N_1;v_1)$ and $G_2 = (N_2;v_2)$; V is a solution of G if and only if it has the following form*: $V = V_1 \times V_2$, *where V_1 and V_2 satisfy*:

(7.4) *V_1 is a solution of $G_1(t)$ and V_2 is a solution of $G_2(-t)$.*

(7.5) *$G_1(t)$ and $G_2(-t)$ have no fully detached i.r.p.v's;*

and

(7.6) $-v_2(N_2) \leqq t \leqq v_1(N_1)$.

Example 7.4 shows that the range of the transfer determined by the kernel may be strictly contained in the range determined by solution theory. We remark that the opposite case does also occur. So, no general conclusions can be made and a detailed investigation is called for.

(16) The determination of the payments by a solution is, of course, generally not unique.
(17) Not necessarily ordinary.

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